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On the matrix difference $\mathbf{I} - \mathbf{A}$

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ABSTRACT

For an $n \times n$ complex matrix $\mathbf{A}$ and the $n \times n$ identity matrix $\mathbf{I}_n$, the difference $\mathbf{I}_n - \mathbf{A}$ is investigated. By exploiting a partitioned representation, several features of such a difference are identified. In particular, expressions for its Moore–Penrose inverse in some specific situations are established, and representations of the pertinent projectors are derived. Special attention is paid to the problem, how certain properties of $\mathbf{A}$ and $\mathbf{I}_n - \mathbf{A}$ are related. The properties in question deal with known classes of matrices, such as GP, EP, partial isometries, bi-EP, normal, projectors, and nilpotent. An important part of the paper is devoted to demonstrating how to obtain representations of orthogonal projectors onto various subspaces determined by $\mathbf{A}$ and/or $\mathbf{I}_n - \mathbf{A}$. Several such representations are provided and a number of relevant conclusions originating from them are identified.

1. Introduction

Let $\mathbb{C}_{n,n}$ be the set of $n \times n$ complex matrices, and let $\mathbf{A} \in \mathbb{C}_{n,n}$. The symbols $\mathbf{A}^*$, $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, and $\text{rk}(\mathbf{A})$ denote conjugate transpose, column space, null space, and rank of $\mathbf{A}$, respectively. Furthermore, $\overline{\mathbf{A}}$ stands for $\overline{\mathbf{A}} = \mathbf{I}_n - \mathbf{A}$, where $\mathbf{I}_n$ is the identity matrix of order n. Customarily, $\mathbf{A}^\dagger$ means the Moore–Penrose inverse of $\mathbf{A}$, i.e., the unique matrix satisfying the equations

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A},\ \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger,\ \mathbf{A}\mathbf{A}^\dagger = (\mathbf{A}\mathbf{A}^\dagger)^*,\ \mathbf{A}^\dagger\mathbf{A} = (\mathbf{A}^\dagger\mathbf{A})^*. \tag{1.1}$$

An essential property of the Moore–Penrose inverse is that it can be used to represent orthogonal projectors. Namely, $\mathbf{A}\mathbf{A}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{A})$, whereas $\mathbf{I}_n - \mathbf{A}\mathbf{A}^\dagger$ is the orthogonal projector onto $\mathcal{N}(\mathbf{A}^*)$, such that $\mathbb{C}_{n,1} = \mathcal{R}(\mathbf{A}) \overset{\perp}{\oplus} \mathcal{N}(\mathbf{A}^*)$, with the symbol $\overset{\perp}{\oplus}$ being used to indicate that the two subspaces involved in the direct sum are orthogonal. Similarly, $\mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{I}_n - \mathbf{A}^\dagger\mathbf{A}$ are the orthogonal projectors onto $\mathcal{R}(\mathbf{A}^*)$ and $\mathcal{N}(\mathbf{A})$, respectively, where $\mathbb{C}_{n,1} = \mathcal{R}(\mathbf{A}^*) \overset{\perp}{\oplus} \mathcal{N}(\mathbf{A})$. In what follows, the symbol $\mathbf{P}_\chi$ denotes the orthogonal projector onto the subspace $\chi \subseteq \mathbb{C}_{n,1}$ and $\mathbf{Q}_\chi = \mathbf{I}_n - \mathbf{P}_\chi$. To shorten the notation, $\mathbf{P}_\mathbf{A}$ and $\mathbf{Q}_\mathbf{A}$ will be used for the orthogonal projectors onto $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^*)$, respectively, i.e., $\mathbf{P}_\mathbf{A} = \mathbf{A}\mathbf{A}^\dagger$ and $\mathbf{Q}_\mathbf{A} = \mathbf{I}_n - \mathbf{A}\mathbf{A}^\dagger$.

A crucial role in the subsequent considerations will be played by the following result given in [1] as Corollary 6.

Lemma 1. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be of rank r. Then there exists unitary $\mathbf{U} \in \mathbb{C}_{n,n}$ such that*

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma K} & \mathbf{\Sigma L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{1.2}$$

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where $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1 \mathbf{I}_{r_1}, \ldots, \sigma_t \mathbf{I}_{r_t})$ *is the diagonal matrix of singular values of* $\mathbf{A}$, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, *and* $\mathbf{K} \in \mathbb{C}_{r,r}$, $\mathbf{L} \in \mathbb{C}_{r,n-r}$ *satisfy*

$$\mathbf{K}\mathbf{K}^* + \mathbf{L}\mathbf{L}^* = \mathbf{I}_r. \tag{1.3}$$

From (1.2), it follows that

$$\mathbf{A}^* = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{L}^*\boldsymbol{\Sigma} & \mathbf{0} \end{pmatrix}\mathbf{U}^* \quad \text{and} \quad \mathbf{A}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\boldsymbol{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{L}^*\boldsymbol{\Sigma}^{-1} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{1.4}$$

Formulae (1.2)–(1.4) can be used to confirm the following characterizations:

(a) $\mathbf{A}$ is GP (i.e., $\mathrm{rk}(\mathbf{A}^2) = \mathrm{rk}(\mathbf{A})$) if and only if $\mathbf{K}$ is nonsingular,
(b) $\mathbf{A}$ is EP (i.e., $\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger\mathbf{A}$) if and only if $\mathbf{L} = \mathbf{0}$,
(c) $\mathbf{A}$ is a partial isometry (i.e., $\mathbf{A}^* = \mathbf{A}^\dagger$) if and only if $\boldsymbol{\Sigma} = \mathbf{I}_r$,
(d) $\mathbf{A}$ is bi-EP (i.e., $\mathbf{A}\mathbf{A}^\dagger\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}^\dagger\mathbf{A}\mathbf{A}\mathbf{A}^\dagger$) if and only if $\mathbf{K}$ is a partial isometry,
(e) $\mathbf{A}$ is normal (i.e., $\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$) if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{K}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{K}$,
(f) $\mathbf{A}$ is an oblique projector (i.e., $\mathbf{A}^2 = \mathbf{A}$) if and only if $\boldsymbol{\Sigma}\mathbf{K} = \mathbf{I}_r$,
(g) $\mathbf{A}$ is an orthogonal projector (i.e., $\mathbf{A}^2 = \mathbf{A} = \mathbf{A}^*$) if and only if $\boldsymbol{\Sigma} = \mathbf{I}_r$, $\mathbf{K} = \mathbf{I}_r$,
(h) $\mathbf{A}$ is Hermitian (i.e., $\mathbf{A}^* = \mathbf{A}$) if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{K}^*\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{K}$,
(i) $\mathbf{A}$ is nilpotent of index 2 (i.e., $\mathbf{A}^2 = \mathbf{0}$) if and only if $\mathbf{K} = \mathbf{0}$.

The first five of these equivalences were established in [1, Corollary 6], the next two are quoted after [2, Lemma 1], the eighth one was given in [3, p. 300], whereas the last one in [4, Lemma 2]. It is noteworthy that the characterizations of EP and normal matrices given in [1, Corollary 6] contain also the requirement that $\mathbf{K}$ is unitary. This requirement is not present in points (b) and (e) above, for in view of (1.3), $\mathbf{L} = \mathbf{0} \Leftrightarrow \mathbf{K}^* = \mathbf{K}^{-1}$. A similar comment concerns characterization of orthogonal projectors which in [2, Lemma 1] has the form: $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{L} = \mathbf{0}$, $\boldsymbol{\Sigma} = \mathbf{I}_r$, $\mathbf{K} = \mathbf{I}_r$. The condition $\mathbf{L} = \mathbf{0}$ is omitted in point (g), for (1.3) ensures that $\mathbf{K} = \mathbf{I}_r \Rightarrow \mathbf{L} = \mathbf{0}$. Several further results involving the representation (1.2) can be found in [5].

Our attention in the paper focuses on the matrices obtained by subtracting a given square matrix from the identity matrix of the same order. The main results are provided in Section 2, where the partitioned representation of matrices introduced in Lemma 1 is utilized to perform extensive investigations over the difference. In particular, expressions for its Moore–Penrose inverse in some specific situations are established, and representations of the orthogonal projectors attributed to it are derived. Special attention is paid to the problem how certain properties of $\mathbf{A}$ and $\overline{\mathbf{A}} = \mathbf{I}_n - \mathbf{A}$ are related. The properties in question deal with known classes of matrices, such as GP, EP, partial isometries, bi-EP, normal, projectors, and nilpotent. An important part of the paper is devoted to demonstrating how to obtain representations of orthogonal projectors onto various subspaces determined by $\mathbf{A}$ and/or $\overline{\mathbf{A}}$. Several such representations are provided and a number of pertinent conclusions are identified.

2. Results

From (1.2), it follows that $\overline{\mathbf{A}} = \mathbf{I}_n - \mathbf{A}$ is of the form

$$\overline{\mathbf{A}} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{G}} & -\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*, \tag{2.1}$$

where $\overline{\mathbf{G}} = \mathbf{I}_r - \mathbf{G} = \mathbf{I}_r - \boldsymbol{\Sigma}\mathbf{K}$ and $\mathbf{H} = \boldsymbol{\Sigma}\mathbf{L}$. According to [6, Definition 3.4.2], the matrix $\overline{\mathbf{A}}$ of the form (2.1) is *properly partitioned* if its Moore–Penrose inverse can be written as

$$\overline{\mathbf{A}}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{0} & \mathbf{F}_{22} \end{pmatrix}\mathbf{U}^*,$$

with $\mathbf{F}_{11} \in \mathbb{C}_{r,r}$ and $\mathbf{F}_{22} \in \mathbb{C}_{n-r,n-r}$. From [6, Theorem 3.4.1], it is seen that $\overline{\mathbf{A}}$ is properly partitioned if and only if $\mathcal{R}(-\mathbf{H}) \subseteq \mathcal{R}(\overline{\mathbf{G}})$ and $\mathcal{R}(-\mathbf{H}^*) \subseteq \mathcal{R}(\mathbf{I}_{n-r})$, which can be reduced to

$$\mathcal{R}(\mathbf{H}) \subseteq \mathcal{R}(\overline{\mathbf{G}}). \tag{2.2}$$

The theorem below provides representations of $\overline{\mathbf{A}}^{\dagger}$ as well as the orthogonal projectors onto $\mathcal{R}(\overline{\mathbf{A}})$ and $\mathcal{R}(\overline{\mathbf{A}}^{*})$ when $\overline{\mathbf{A}}$ is properly partitioned.

Theorem 1. *Let $\overline{\mathbf{A}}$ of the form (2.1) be properly partitioned. Then:*

(i) $\overline{\mathbf{A}}^{\dagger} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{G}}^{\dagger} & \overline{\mathbf{G}}^{\dagger}\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}$,

(ii) $\mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}$,

(iii) $\mathbf{P}_{\overline{\mathbf{A}}^{*}} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}^{*}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}$.

Proof. The expression given in point (i) is a direct consequence of [6, Theorem 3.4.1]. The formulae for $\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}}\,\overline{\mathbf{A}}^{\dagger}$ and $\mathbf{P}_{\overline{\mathbf{A}}^{*}} = \overline{\mathbf{A}}^{\dagger}\overline{\mathbf{A}}$ follow from (2.1), point (i) of the theorem, and the fact that $\mathbf{P}_{\overline{\mathbf{G}}}\mathbf{H} = \mathbf{H}$, which is an equivalent form of (2.2). □

By virtue of [7, Corollary 19.1], it follows from (2.1) that

$$\mathrm{rk}(\overline{\mathbf{A}}) = \mathrm{rk}(\overline{\mathbf{G}}) + n - r, \tag{2.3}$$

whence it is seen that $\overline{\mathbf{A}}$ is nonsingular if and only if $\overline{\mathbf{G}}$ is nonsingular. The inclusion (2.2) shows that nonsingularity of $\overline{\mathbf{G}}$ is sufficient for $\overline{\mathbf{A}}$ to be properly partitioned.

In the context of Theorem 1, the question arises under what conditions $\overline{\mathbf{A}}$ is properly partitioned. Two such situations are when $\mathbf{A}$ is either EP (i.e., $\mathbf{L} = \mathbf{0}$) or, simultaneously, $\mathbf{A}$ is a partial isometry (i.e., $\mathbf{\Sigma} = \mathbf{I}_r$) and $\mathbf{K}$ is Hermitian. In the latter case, we have

$$\begin{aligned}\mathcal{R}(\mathbf{H}) &= \mathcal{R}(\mathbf{L}) = \mathcal{R}(\mathbf{L}\mathbf{L}^{*}) = \mathcal{R}(\mathbf{I}_r - \mathbf{K}\mathbf{K}^{*}) = \mathcal{R}(\mathbf{I}_r - \mathbf{K}^{2}) \\ &= \mathcal{R}[(\mathbf{I}_r - \mathbf{K})(\mathbf{I}_r + \mathbf{K})] \subseteq \mathcal{R}(\mathbf{I}_r - \mathbf{K}) = \mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma}\mathbf{K}) = \mathcal{R}(\overline{\mathbf{G}}).\end{aligned}$$

It can be directly verified that when $\mathrm{rk}(\overline{\mathbf{A}}) = n$, then

$$\overline{\mathbf{A}}^{-1} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{G}}^{-1} & \overline{\mathbf{G}}^{-1}\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}.$$

Furthermore, the matrix $\overline{\mathbf{A}}^{-1}$ exists if and only if $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\overline{\mathbf{A}})$, or, in other words, $\mathbf{P}_{\overline{\mathbf{A}}}\mathbf{A} = \mathbf{A}$. This fact can be concluded from the following equivalences

$$\mathbf{P}_{\overline{\mathbf{A}}}\mathbf{A} = \mathbf{A} \Leftrightarrow \mathbf{P}_{\overline{\mathbf{A}}}(\overline{\mathbf{A}} - \mathbf{I}_n) = -\mathbf{A} \Leftrightarrow \overline{\mathbf{A}} - \mathbf{P}_{\overline{\mathbf{A}}} = -\mathbf{A} \Leftrightarrow \mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{I}_n.$$

Note that $\overline{\mathbf{A}}$ is nonsingular whenever $\mathbf{A}$ is nilpotent of index 2 (i.e., $\mathbf{K} = \mathbf{0}$), in which case $\overline{\mathbf{A}}^{-1} = \mathbf{I}_n + \mathbf{A}$.

In the general situation, when $\overline{\mathbf{A}}$ is not necessarily nonsingular, the formula for $\overline{\mathbf{A}}^{\dagger}$ is rather difficult to establish. However, by exploiting the identity $\overline{\mathbf{A}}^{\dagger} = \overline{\mathbf{A}}^{*}(\overline{\mathbf{A}}\,\overline{\mathbf{A}}^{*})^{\dagger}$, in which $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^{*}$ is a nonnegative definite matrix of the form

$$\overline{\mathbf{A}}\,\overline{\mathbf{A}}^{*} = \mathbf{U}\begin{pmatrix} \mathbf{F} & -\mathbf{H} \\ -\mathbf{H}^{*} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}, \tag{2.4}$$

where $\mathbf{F} = \overline{\mathbf{G}}\,\overline{\mathbf{G}}^{*} + \mathbf{H}\mathbf{H}^{*}$, the calculations can be distinctly simplified. The Moore–Penrose inverse of the matrix (2.4) may be obtained on account of [8, Theorem 1] by doing straightforward, but rather time-consuming, calculations. (Note that the results given in [8] concern matrices of real entries, but an extension to the complex case follows directly.)

Inspired by the investigations in [8], we apply the present approach to a special case considered therein. For this reason, let $\mathbf{S} = \mathbf{I}_{n-r} - \mathbf{H}^{*}\mathbf{F}^{\dagger}\mathbf{H}$ and assume that condition (ii) in [8, Corollary 2] is satisfied, i.e.,

$$(\mathbf{I}_{n-r} - \mathbf{S}^{\dagger}\mathbf{S})\mathbf{H}^{*}\mathbf{F}^{\dagger} = \mathbf{0}, \tag{2.5}$$

which can be equivalently expressed as $\mathcal{R}(\mathbf{H}^{*}\mathbf{F}^{\dagger}) \subseteq \mathcal{R}(\mathbf{S})$. Moreover, from [8, p. 116] we conclude that each of the rank conditions $\mathrm{rk}(\mathbf{S}) = n - r$ and $\mathrm{rk}(\overline{\mathbf{A}}) = \mathrm{rk}(\mathbf{F}) + n - r$ also holds if and only if (2.5) is satisfied. The next theorem establishes formulae for the orthogonal projectors onto the column spaces of $\overline{\mathbf{A}}$ and $\overline{\mathbf{A}}^{*}$ provided that (2.5) holds.

Theorem 2. *Let $\overline{\mathbf{A}}$ of the form (2.1) be such that (2.5) holds. Then:*

(i) $\mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{F}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}$,

(ii) $\mathbf{P}_{\overline{\mathbf{A}}^{*}} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{F}^{*}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^{*}$,

where $\mathbf{F} = \overline{\mathbf{G}}\,\overline{\mathbf{G}}^{*} + \mathbf{H}\mathbf{H}^{*}$.

Proof. First note that [8, Corollary 2] leads to the conclusion that the Moore–Penrose inverse of $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*$ of the form (2.4) is given by

$$(\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*)^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{F}^\dagger + \mathbf{F}^\dagger\mathbf{H}\mathbf{S}^{-1}\mathbf{H}^*\mathbf{F}^\dagger & \mathbf{F}^\dagger\mathbf{H}\mathbf{S}^{-1} \\ \mathbf{S}^{-1}\mathbf{H}^*\mathbf{F}^\dagger & \mathbf{S}^{-1} \end{pmatrix}\mathbf{U}^*, \tag{2.6}$$

with nonsingularity of $\mathbf{S}$ ensured by (2.5). Now using $\mathbf{S} = \mathbf{I}_{n-r} - \mathbf{H}^*\mathbf{F}^\dagger\mathbf{H}$ and the fact that $\mathbf{P_F}\mathbf{H} = \mathbf{H}$ (see [9, Theorem 1]), formulae (2.4) and (2.6) entail

$$\mathbf{P}_{\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*} = \mathbf{U}\begin{pmatrix} \mathbf{P_F} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{2.7}$$

Since $\mathcal{R}(\mathbf{P}_{\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*}) = \mathcal{R}(\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*) = \mathcal{R}(\overline{\mathbf{A}})$, it follows that the projector $\mathbf{P}_{\overline{\mathbf{A}}}$ is of the form (2.7), and the proof of point (i) is complete.

The representation provided in point (ii) is established similarly. □

Substituting (2.1) and (2.6) to $\overline{\mathbf{A}}^\dagger = \overline{\mathbf{A}}^*(\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*)^\dagger$ shows that the Moore–Penrose inverse of $\overline{\mathbf{A}}$ under the condition (2.5) is given by

$$\overline{\mathbf{A}}^\dagger = \mathbf{U}\begin{pmatrix} \overline{\mathbf{G}}^*\mathbf{F}^\dagger + \overline{\mathbf{G}}^*\mathbf{F}^\dagger\mathbf{H}\mathbf{S}^{-1}\mathbf{H}^*\mathbf{F}^\dagger & \overline{\mathbf{G}}^*\mathbf{F}^\dagger\mathbf{H}\mathbf{S}^{-1} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{2.8}$$

The representation (2.8) can be used to establish an alternative proof of Theorem 2.

Theorems 1 and 2 given above provide representations of the projectors $\mathbf{P}_{\overline{\mathbf{A}}}$ and $\mathbf{P}_{\overline{\mathbf{A}}^*}$ when $\overline{\mathbf{A}}$, or, in fact $\mathbf{A}$, has some additional property. In the next theorem we establish formulae for these projectors in the general case. First, let us focus our attention on the so-called g-inverses, which are obtained as solutions to the first equation in (1.1). The lemma below delivers a formula for a g-inverse of a particular upper block triangular matrix.

Lemma 2. *Let* $\mathbf{T} \in \mathbb{C}_{n,n}$ *be of the form*

$$\mathbf{T} = \begin{pmatrix} \mathbf{V} & \mathbf{W} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}.$$

Then a g*-inverse of* $\mathbf{T}$ *is*

$$\mathbf{T}^- = \begin{pmatrix} \mathbf{V}^- & -\mathbf{V}^-\mathbf{W} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix},$$

where $\mathbf{V}^-$ *is a* g*-inverse of* $\mathbf{V}$.

Proof. The assertion is established by direct verification of the condition $\mathbf{T}\mathbf{T}^-\mathbf{T} = \mathbf{T}$. □

By applying Lemma 2 to the matrix given in (2.1) we arrive at the representation of a g-inverse of $\overline{\mathbf{A}}$ in the form

$$\overline{\mathbf{A}}^- = \mathbf{U}\begin{pmatrix} \overline{\mathbf{G}}^- & \overline{\mathbf{G}}^-\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*, \tag{2.9}$$

where $\overline{\mathbf{G}}^-$ is a g-inverse of $\overline{\mathbf{G}}$. Furthermore, if we replace a non-unique g-inverse $\overline{\mathbf{G}}^-$ with the unique Moore–Penrose inverse $\overline{\mathbf{G}}^\dagger$, then (2.1) and (2.9) lead to the following representation of an oblique projector onto the column space of $\overline{\mathbf{A}}$

$$\overline{\mathbf{A}}\,\overline{\mathbf{A}}^- = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}} & -\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^* := \mathbf{M}, \tag{2.10}$$

where $\mathbf{Q}_{\overline{\mathbf{G}}} = \mathbf{I}_r - \mathbf{P}_{\overline{\mathbf{G}}}$. Since $\mathcal{R}(\overline{\mathbf{A}}) = \mathcal{R}(\mathbf{M})$, it follows that $\mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{P}_{\mathbf{M}} = \mathbf{P}_{\overline{\mathbf{A}}\,\overline{\mathbf{A}}^-}$, which means that we can exploit the matrix $\mathbf{M}$ defined in (2.10) to derive a general formula for $\mathbf{P}_{\overline{\mathbf{A}}}$. The formula, as well as the representation of $\mathbf{P}_{\overline{\mathbf{A}}^*}$, is given in what follows.

Theorem 3. *Let* $\overline{\mathbf{A}}$ *be of the form* (2.1)*. Then:*

$$\text{(i) } \mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}} + \mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{N}^{-1}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}} & -\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{N}^{-1} \\ -\mathbf{N}^{-1}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}} & \mathbf{N}^{-1} \end{pmatrix}\mathbf{U}^*, \tag{2.11}$$

where $\mathbf{N} = \mathbf{I}_{n-r} + \mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}$,

$$\text{(ii) } \mathbf{P}_{\overline{\mathbf{A}}^*} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}^*} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{2.12}$$

Proof. From (2.10) it follows that $\mathbf{M}^\dagger = (\mathbf{M}^*\mathbf{M})^\dagger\mathbf{M}^*$ takes the from

$$\mathbf{M}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}} & \mathbf{0} \\ -\mathbf{N}^{-1}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}} & \mathbf{N}^{-1} \end{pmatrix}\mathbf{U}^*, \tag{2.13}$$

where $\mathbf{N} = \mathbf{I}_{n-r} + \mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}$ is a sum of two Hermitian matrices one of which is positive definite and the other is nonnegative definite, which means that $\mathbf{N}$ is nonsingular. Premultiplying (2.13) by $\mathbf{M}$ leads to the representation of $\mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{P}_{\mathbf{M}}$ claimed in the theorem.

From (2.9) we conclude that a g-inverse of $\overline{\mathbf{A}}^*$ has the form

$$(\overline{\mathbf{A}}^*)^- = \mathbf{U}\begin{pmatrix} (\overline{\mathbf{G}}^*)^\dagger & \mathbf{0} \\ \mathbf{H}^*(\overline{\mathbf{G}}^*)^\dagger & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*.$$

Hence, by (2.1),

$$\overline{\mathbf{A}}^*(\overline{\mathbf{A}}^*)^- = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}^*} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{2.14}$$

It is seen that (2.14) is the orthogonal projector onto $\mathcal{R}[\overline{\mathbf{A}}^*(\overline{\mathbf{A}}^*)^-] = \mathcal{R}(\overline{\mathbf{A}}^*)$, which establishes point (ii) of the theorem. □

Note that, when $\overline{\mathbf{A}}$ of the form (2.1) is properly partitioned, i.e., inclusion (2.2) holds, then the representation (2.11) reduces to the block diagonal matrix given in point (ii) of Theorem 1. On the other hand, the representation given in point (ii) of Theorem 3 coincides with the one provided in point (iii) of Theorem 1 which means that $\mathbf{P}_{\overline{\mathbf{A}}^*}$ has the same form regardless whether $\overline{\mathbf{A}}$ is properly partitioned or not.

Theorem 3 allows to obtain a representation of the orthogonal projectors onto the null spaces of $\overline{\mathbf{A}}$ and $\overline{\mathbf{A}}^*$. For example, from (2.12) it is seen that the orthogonal projector onto $\mathcal{N}(\overline{\mathbf{A}})$, given by $\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})} = \mathbf{I}_n - \mathbf{P}_{\overline{\mathbf{A}}^*}$, is of the form

$$\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{G}}^*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{2.15}$$

It is of interest to inquire what properties of the matrix $\mathbf{A}$ are inherited by $\overline{\mathbf{A}}$. Let us have a look at the nine equivalences (a)–(i) listed below Lemma 1. It can easily be verified that $\mathbf{A}$ is normal if and only if so is $\overline{\mathbf{A}}$. Similar statements are also valid when $\mathbf{A}$ is an oblique projector, an orthogonal projector, and Hermitian. However, the remaining five equivalences, i.e., (a)–(d) and (i), are no longer valid when $\mathbf{A}$ is replaced with $\overline{\mathbf{A}}$. To confirm this observation, let us assume $\mathbf{A}$ to be of the form

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad \text{where } a, b, c \in \mathbb{C}. \tag{2.16}$$

By taking $a = 1, b \neq 0, c = 1$ we obtain from (2.16) a matrix which is GP and EP, but $\overline{\mathbf{A}}$ is neither GP nor EP; by taking $a = 2, b \neq 0, c = 1$ we get a matrix which is bi-GP, but $\overline{\mathbf{A}}$ is not bi-EP; and by taking $a = 0, c = 0$ we arrive at a matrix which is nilpotent of index 2 for any b, but $\overline{\mathbf{A}}$ does not have this property. To show that when $\mathbf{A}$ is a partial isometry, then $\overline{\mathbf{A}}$ need not be so, consider $\mathbf{A} = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$. Actually, as can be verified with the use of (2.1), $\overline{\mathbf{A}}$ is nilpotent of index 2 only when $\mathbf{A}$ is nonsingular, in which case $\overline{\mathbf{A}}$ is nilpotent of index 2 if and only if $\overline{\mathbf{G}}$ is nilpotent of index 2.

The next theorem identifies conditions necessary and sufficient for $\overline{\mathbf{A}}$ to be GP, EP, a partial isometry, and bi-EP.

Theorem 4. *Let $\overline{\mathbf{A}}$ be of the form (2.1). Then:*

(i) *$\overline{\mathbf{A}}$ is GP if and only if $\overline{\mathbf{G}}$ is GP,*
(ii) *$\overline{\mathbf{A}}$ is EP if and only if $\overline{\mathbf{A}}$ is properly partitioned and $\overline{\mathbf{G}}$ is EP,*
(iii) *$\overline{\mathbf{A}}$ is a partial isometry if and only if $\mathbf{A}$ is EP and $\overline{\mathbf{G}}$ is a partial isometry,*
(iv) *$\overline{\mathbf{A}}$ is bi-EP if and only if $\overline{\mathbf{G}}$ is bi-EP and $\mathcal{R}(\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}) \subseteq \mathcal{R}(\overline{\mathbf{G}}^*)$.*

Proof. From (2.1), we have

$$\overline{\mathbf{A}}^2 = \mathbf{U}\begin{pmatrix} \overline{\mathbf{G}}^2 & -\overline{\mathbf{G}}\mathbf{H} - \mathbf{H} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*,$$

whence, on account of [7, Corollary 19.1], we get $\mathrm{rk}(\overline{\mathbf{A}}^2) = \mathrm{rk}(\overline{\mathbf{G}}^2) + n - r$. Combining this identity with (2.3) shows that $\mathrm{rk}(\overline{\mathbf{A}}^2) = \mathrm{rk}(\overline{\mathbf{A}}) \Leftrightarrow \mathrm{rk}(\overline{\mathbf{G}}^2) = \mathrm{rk}(\overline{\mathbf{G}})$, which constitutes point (i) of the theorem.

Recall that $\overline{\mathbf{A}}$ is EP if and only if $\mathbf{P}_{\overline{\mathbf{A}}} = \mathbf{P}_{\overline{\mathbf{A}}^*}$. From Theorem 3, it follows that this equality can be equivalently expressed as the conjunction $\mathbf{P}_{\overline{\mathbf{G}}} = \mathbf{P}_{\overline{\mathbf{G}}^*}$ and $\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H} = \mathbf{0}$. The first of these conditions means that $\overline{\mathbf{G}}$ is EP, whereas the second is equivalent to (2.2), i.e., to the requirement that $\overline{\mathbf{A}}$ is properly partitioned.

The fact that $\overline{\mathbf{A}}$ is a partial isometry can be alternatively written as $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*\overline{\mathbf{A}} = \overline{\mathbf{A}}$, and from (2.1) we get

$$\overline{\mathbf{A}}\,\overline{\mathbf{A}}^*\overline{\mathbf{A}} = \overline{\mathbf{A}} \Leftrightarrow \mathbf{H} = \mathbf{0} \quad \text{and} \quad \overline{\mathbf{G}}\,\overline{\mathbf{G}}^*\overline{\mathbf{G}} = \overline{\mathbf{G}}.$$

Since $\mathbf{H} = \mathbf{0} \Leftrightarrow \mathbf{\Sigma L} = \mathbf{0} \Leftrightarrow \mathbf{L} = \mathbf{0}$, which is equivalent to the requirement that $\mathbf{A}$ is EP, point (iii) of the theorem is established.

From Theorem 3, it follows that $\overline{\mathbf{A}}$ is bi-EP, i.e., satisfies $\mathbf{P}_{\overline{\mathbf{A}}}\mathbf{P}_{\overline{\mathbf{A}}^*} = \mathbf{P}_{\overline{\mathbf{A}}^*}\mathbf{P}_{\overline{\mathbf{A}}}$, if and only if

$$\mathbf{P}_{\overline{\mathbf{G}}}\mathbf{P}_{\overline{\mathbf{G}}^*} + \mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{N}^{-1}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{P}_{\overline{\mathbf{G}}^*} = \mathbf{P}_{\overline{\mathbf{G}}^*}\mathbf{P}_{\overline{\mathbf{G}}} + \mathbf{P}_{\overline{\mathbf{G}}^*}\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{N}^{-1}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}} \tag{2.17}$$

and

$$\mathbf{P}_{\overline{\mathbf{G}}^*}\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H} = \mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}. \tag{2.18}$$

In the light of (2.18), equality (2.17) reduces to $\mathbf{P}_{\overline{\mathbf{G}}}\mathbf{P}_{\overline{\mathbf{G}}^*} = \mathbf{P}_{\overline{\mathbf{G}}^*}\mathbf{P}_{\overline{\mathbf{G}}}$. Since (2.18) can be equivalently expressed as $\mathcal{R}(\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}) \subseteq \mathcal{R}(\mathbf{P}_{\overline{\mathbf{G}}^*}) = \mathcal{R}(\overline{\mathbf{G}}^*)$, the last point of the theorem follows. □

Note that point (iii) of Theorem 4 is related to Lemma 3.3 in [10], which claims that when $\mathbf{A}$ is a contraction, then $\overline{\mathbf{A}}$ is EP. A particular version of this point is given in the next result, which refers to a clear fact that when a partial isometry is nonsingular, then it is unitary.

Theorem 5. *Let $\mathbf{A}$ be of the form (1.2). Then $\overline{\mathbf{A}}$ is unitary if and only if $\mathbf{A}$ is EP and $\mathbf{\Sigma}^2 = \mathbf{K}^{-1}\mathbf{\Sigma} + \mathbf{\Sigma K}$.*

Proof. From (1.2), it follows that $\overline{\mathbf{A}}\,\overline{\mathbf{A}}^* = \mathbf{I}_n$ if and only if $\mathbf{L} = \mathbf{0}$ and $\mathbf{\Sigma}^2 = \mathbf{K}^*\mathbf{\Sigma} + \mathbf{\Sigma K}$. The former of these conditions means that $\mathbf{A}$ is EP, whereas the latter, since $\mathbf{L} = \mathbf{0} \Leftrightarrow \mathbf{K}^* = \mathbf{K}^{-1}$, can be expressed in the form claimed in the theorem. □

Subsequently we provide formulae for the orthogonal projectors onto various subspaces determined by $\mathbf{A}$ and $\overline{\mathbf{A}}$. A crucial role in those derivations will be played by the following lemma.

Lemma 3. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ be orthogonal projectors. Then:*

(i) $\mathbf{P} + \overline{\mathbf{P}}(\overline{\mathbf{P}}\mathbf{Q})^\dagger$ *is the orthogonal projector onto $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$,*
(ii) $\mathbf{P} - \mathbf{P}(\mathbf{P}\overline{\mathbf{Q}})^\dagger$ *is the orthogonal projector onto $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$.*

Proof. Statements (i) and (ii) constitute equivalences (3.1) ⇔ (3.6) and (4.1) ⇔ (4.8) in [11], respectively. □

Using Lemma 3 we obtain the following representations of the orthogonal projectors onto sum and intersection of $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\overline{\mathbf{A}}^*)$.

Theorem 6. *Let $\mathbf{A}$ be of the form (1.2). Then:*

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{A})+\mathcal{R}(\overline{\mathbf{A}}^*)} = \mathbf{I}_n$,

(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}}^*)} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{G}}^*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$,

where $\mathbf{G} = \mathbf{\Sigma K}$.

Proof. From (1.2) and (1.4), it follows that

$$\mathbf{P}_{\mathbf{A}} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{2.19}$$

Applying point (i) of Lemma 3 to (2.12) and (2.19) directly leads to the conclusion that the orthogonal projector onto the sum of $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\overline{\mathbf{A}}^*)$ is the identity matrix. Similarly, the representations (2.12) and (2.19) substituted to point (ii) of Lemma 3 entail the representation given in point (ii) of the theorem. □

The identity in point (i) of Theorem 6 is equivalent to the equality $\mathcal{R}(\mathbf{A}) + \mathcal{R}(\overline{\mathbf{A}}^*) = \mathbb{C}_{n,1}$, whose validity can also be seen by exploiting the inclusion $\mathcal{N}(\overline{\mathbf{A}}) \subseteq \mathcal{R}(\mathbf{A})$, which is always satisfied, and which can alternatively be expressed in terms of orthogonal complements as $\mathcal{R}(\mathbf{A})^\perp \subseteq \mathcal{N}(\overline{\mathbf{A}})^\perp$. Hence, we have $\mathbb{C}_{n,1} = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{A})^\perp \subseteq \mathcal{R}(\mathbf{A}) + \mathcal{N}(\overline{\mathbf{A}})^\perp = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\overline{\mathbf{A}}^*)$, from where the aforementioned equality follows.

Point (ii) of Theorem 6 yields $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\overline{\mathbf{A}}^*) = \{\mathbf{0}\} \Leftrightarrow \mathbf{G} = \mathbf{I}_r$. This is equivalent to $\mathbf{\Sigma K} = \mathbf{I}_r$ which is a necessary and sufficient condition for $\mathbf{A}$ to be an oblique projector. Interestingly, the equivalence between idempotency of $\mathbf{A}$ and the requirement that $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\overline{\mathbf{A}}^*) = \{\mathbf{0}\}$ seems to be in a contrast to [12, Theorem 1], according to which $\mathbf{A}^2 = \mathbf{A}$ if and only if $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\overline{\mathbf{A}}) = \{\mathbf{0}\}$ (after Theorem 7 given below we show that $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\overline{\mathbf{A}}) = \{\mathbf{0}\}$ is equivalent to $\mathrm{rk}(\overline{\mathbf{A}}) + \mathrm{rk}(\mathbf{A}) = n$, which dissolves this contrast). Note also that $\mathcal{R}(\mathbf{A}) \cap \mathcal{R}(\overline{\mathbf{A}}^*) = \{\mathbf{0}\}$ if and only if $\mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\overline{\mathbf{A}}^*}$ is an orthogonal projector, in which case $\mathbf{P_A}+\mathbf{P}_{\overline{\mathbf{A}}^*}$ projects onto $\mathcal{R}(\mathbf{A})\oplus\mathcal{R}(\overline{\mathbf{A}}^*)$ (along $\mathcal{N}(\mathbf{A})\cap\mathcal{N}(\overline{\mathbf{A}}^*)$); see Theorem in [13, Section 42]. Another comment originating from Theorem in [13, Section 42] is that an alternative proof of point (ii) of the theorem can be based on the fact that when two, possibly oblique, projectors are commuting, then their product is the projector onto the intersection of the column spaces of these projectors (along the sum of their null spaces). Moreover, from $\mathbf{P_A}\mathbf{P}_{\overline{\mathbf{A}}^*}=\mathbf{P}_{\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}}^*)}$ we obtain $\mathbf{P}_{\mathbf{A}^*}\mathbf{P}_{\overline{\mathbf{A}}}=\mathbf{P}_{\mathcal{R}(\mathbf{A}^*)\cap\mathcal{R}(\overline{\mathbf{A}})}$. Additionally, observe that $\dim[\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}}^*)]=\dim[\mathcal{R}(\mathbf{A}^*)\cap\mathcal{R}(\overline{\mathbf{A}})]=\mathrm{rk}(\overline{\mathbf{G}})$.

By combining (2.12), (2.19), and point (ii) of Theorem 6, we arrive at $\mathbf{P}_{\overline{\mathbf{A}}^*}=\mathbf{Q_A}+\mathbf{P}_{\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}}^*)}$, which entails $\mathcal{R}(\overline{\mathbf{A}}^*)=\mathcal{N}(\mathbf{A}^*)\overset{\perp}{\oplus}[\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}}^*)]$.

It is known that $\mathcal{R}(\mathbf{A})+\mathcal{R}(\overline{\mathbf{A}})=\mathbb{C}_{n,1}$, i.e., $\mathbf{P}_{\mathcal{R}(\mathbf{A})+\mathcal{R}(\overline{\mathbf{A}})}=\mathbf{I}_n$. The next theorem provides a formula for the orthogonal projector onto the intersection of $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\overline{\mathbf{A}})$.

Theorem 7. *Let* $\mathbf{A}$ *be of the form* (1.2). *Then*

$$\mathbf{P}_{\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})}=\mathbf{U}\begin{pmatrix}\mathbf{Q_R} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

where $\mathbf{R}=\mathbf{Q}_{\overline{\mathbf{G}}}(\mathbf{I}_r-\mathbf{H}\mathbf{N}^{-1}\mathbf{H}^*)\mathbf{Q}_{\overline{\mathbf{G}}}$, $\mathbf{N}=\mathbf{I}_{n-r}+\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}$, $\mathbf{G}=\mathbf{\Sigma K}$, *and* $\mathbf{H}=\mathbf{\Sigma L}$.

Proof. Inserting (2.11) and (2.19) in point (ii) of Lemma 3 shows that to determine the formula for the orthogonal projector onto $\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})$ we need the Moore–Penrose inverse of

$$\mathbf{P_A}\mathbf{Q}_{\overline{\mathbf{A}}}=\mathbf{U}\begin{pmatrix}\mathbf{R} & \mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{N}^{-1}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{2.20}$$

where $\mathbf{R}$ and $\mathbf{N}$ are as specified in the theorem. By virtue of $(\mathbf{P_A}\mathbf{Q}_{\overline{\mathbf{A}}})^\dagger=\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{P_A}(\mathbf{P_A}\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{P_A})^\dagger$, the representations (2.19) and (2.20) entail

$$(\mathbf{P_A}\mathbf{Q}_{\overline{\mathbf{A}}})^\dagger=\mathbf{U}\begin{pmatrix}\mathbf{P_R} & \mathbf{0}\\ \mathbf{N}^{-1}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{R}^\dagger & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

whence the asserted formula follows by point (ii) of Lemma 3. □

Note that $\mathbf{R}$ specified in Theorem 7 satisfies $(\mathbf{I}_r+\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}})\mathbf{R}=\mathbf{Q}_{\overline{\mathbf{G}}}$, whence $\mathbf{R}=(\mathbf{I}_r+\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}})^{-1}\mathbf{Q}_{\overline{\mathbf{G}}}$. In consequence, $\mathrm{rk}(\mathbf{R})=\mathrm{rk}(\mathbf{Q}_{\overline{\mathbf{G}}})=r-\mathrm{rk}(\overline{\mathbf{G}})$, and Theorem 7 yields $\dim[\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})]=r-\mathrm{rk}(\mathbf{R})=\mathrm{rk}(\overline{\mathbf{G}})$. Combining this observation with (2.3) leads to $\dim[\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})]=\mathrm{rk}(\overline{\mathbf{A}})-n+r$. Thus, $\dim[\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})]=0$ if and only if $\mathrm{rk}(\overline{\mathbf{A}})+\mathrm{rk}(\mathbf{A})=n$, or, in other words, $\mathbf{A}^2=\mathbf{A}$; see [12, Theorem 2].

Another interesting result is obtained from formula (2.2a) in [14], which claims that $\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}})=\mathrm{rk}(\mathbf{A})+\mathrm{rk}(\overline{\mathbf{A}})-n$. In view of the above, we obtain

$$\dim[\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})]=\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}}). \tag{2.21}$$

This identity can alternatively be obtained by virtue of Theorem 2.6 in [15], from which we have

$$\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}})=\mathrm{rk}(\mathbf{A})-\dim[\mathcal{R}(\mathbf{A})\cap\mathcal{N}(\overline{\mathbf{A}})]=r-\dim[\mathcal{N}(\overline{\mathbf{A}})], \tag{2.22}$$

with the last equality originating from the fact that $\mathcal{N}(\overline{\mathbf{A}})\subseteq\mathcal{R}(\mathbf{A})$. Since (2.15) leads to $\dim[\mathcal{N}(\overline{\mathbf{A}})]=r-\mathrm{rk}(\overline{\mathbf{G}})$, from (2.22) we get $\mathrm{rk}(\mathbf{A}\overline{\mathbf{A}})=\mathrm{rk}(\overline{\mathbf{G}})$, which combined with Theorem 7 gives (2.21).

It is clear that $\mathbf{R}$ specified in Theorem 7 satisfies $\mathcal{R}(\mathbf{R})\subseteq\mathcal{R}(\mathbf{Q}_{\overline{\mathbf{G}}})$. By $\mathrm{rk}(\mathbf{R})=\mathrm{rk}(\mathbf{Q}_{\overline{\mathbf{G}}})$, the latter inclusion can be strengthened to the equality $\mathcal{R}(\mathbf{R})=\mathcal{R}(\mathbf{Q}_{\overline{\mathbf{G}}})$, from where we get $\mathbf{Q_R}=\mathbf{P}_{\overline{\mathbf{G}}}$, i.e.,

$$\mathbf{P}_{\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})}=\mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{G}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*.$$

An additional comment is that $\mathbf{A}\overline{\mathbf{A}}=\overline{\mathbf{A}}\mathbf{A}$ implies $\mathcal{R}(\mathbf{A}\overline{\mathbf{A}})\subseteq\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})$. Taking into account that the dimensions of both $\mathcal{R}(\mathbf{A}\overline{\mathbf{A}})$ and $\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})$ are equal, we arrive at $\mathcal{R}(\mathbf{A}\overline{\mathbf{A}})=\mathcal{R}(\mathbf{A})\cap\mathcal{R}(\overline{\mathbf{A}})$.

Lemma 3 allows to obtain formulae for orthogonal projectors practically onto any sum and intersection of column and/or null spaces of $\mathbf{A}$, $\overline{\mathbf{A}}$, and their conjugate transposes. In consequence, several characteristics of those onto spaces, such as, for instance, dimension, can be expressed in terms of matrices $\mathbf{\Sigma}$, $\mathbf{K}$, and $\mathbf{L}$ involved in the representation (1.2). Additional two formulae for projectors are established in the following.

Theorem 8. *Let* $\mathbf{A}$ *be of the form* (1.2). *Then:*

(i) $$\mathbf{P}_{\mathcal{N}(\mathbf{A})\cap\mathcal{N}(\overline{\mathbf{A}})}=\mathbf{U}\begin{pmatrix}\mathbf{Q}_{\overline{\mathbf{G}}^*}-\mathbf{P}_{\mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*,$$

where $\mathbf{G} = \mathbf{\Sigma K}$,

(ii) $$\mathbf{P}_{\mathcal{R}(\overline{\mathbf{A}})\cap\mathcal{N}(\overline{\mathbf{A}})} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{G}}^*} - \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{S}^*(\mathbf{S}\mathbf{Q}_{\overline{\mathbf{G}}^*})^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

where $\mathbf{S} = \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{Q}_{\overline{\mathbf{G}}}(\mathbf{I}_r - \mathbf{H}\mathbf{N}^{-1}\mathbf{H}^*)\mathbf{Q}_{\overline{\mathbf{G}}}$, $\mathbf{N} = \mathbf{I}_{n-r} + \mathbf{H}^*\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}$, $\mathbf{G} = \mathbf{\Sigma K}$, *and* $\mathbf{H} = \mathbf{\Sigma L}$.

Proof. From (1.2) and (1.4), it follows that $\mathbf{P}_{\mathcal{N}(\mathbf{A})} = \mathbf{Q}_{\mathbf{A}^*}$ is of the form

$$\mathbf{P}_{\mathcal{N}(\mathbf{A})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \mathbf{K}^*\mathbf{K} & -\mathbf{K}^*\mathbf{L} \\ -\mathbf{L}^*\mathbf{K} & \mathbf{I}_{n-r} - \mathbf{L}^*\mathbf{L} \end{pmatrix}\mathbf{U}^*. \tag{2.23}$$

Applying point (ii) of Lemma 3 to (2.15) and (2.23) shows that to derive a formula for $\mathbf{P}_{\mathcal{N}(\mathbf{A})\cap\mathcal{N}(\overline{\mathbf{A}})}$ we need the Moore–Penrose inverse of

$$\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{P}_{\mathbf{A}^*} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*\mathbf{K} & \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{2.24}$$

By virtue of $(\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{P}_{\mathbf{A}^*})^\dagger = \mathbf{P}_{\mathbf{A}^*}\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}(\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{P}_{\mathbf{A}^*}\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})})^\dagger$, from (2.24) we get

$$(\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{P}_{\mathbf{A}^*})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{K}^*\mathbf{K}\mathbf{Q}_{\overline{\mathbf{G}}^*}(\mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*\mathbf{K}\mathbf{Q}_{\overline{\mathbf{G}}^*})^\dagger & \mathbf{0} \\ \mathbf{L}^*\mathbf{K}\mathbf{Q}_{\overline{\mathbf{G}}^*}(\mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*\mathbf{K}\mathbf{Q}_{\overline{\mathbf{G}}^*})^\dagger & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

whence point (ii) of Lemma 3 gives

$$\mathbf{P}_{\mathcal{N}(\mathbf{A})\cap\mathcal{N}(\overline{\mathbf{A}})} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{G}}^*} - \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*\mathbf{K}\mathbf{Q}_{\overline{\mathbf{G}}^*}(\mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*\mathbf{K}\mathbf{Q}_{\overline{\mathbf{G}}^*})^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

which can be simplified to the form provided in point (i) of the theorem.

The second representation given in the theorem is established analogously. On account of (2.11) and (2.15), from point (ii) of Lemma 3 it follows that in order to obtain a formula for $\mathbf{P}_{\mathcal{R}(\overline{\mathbf{A}})\cap\mathcal{N}(\overline{\mathbf{A}})}$, we need the Moore–Penrose inverse of

$$\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{Q}_{\overline{\mathbf{A}}} = \mathbf{U}\begin{pmatrix} \mathbf{S} & \mathbf{T} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{2.25}$$

with $\mathbf{S}$ as specified in the theorem and $\mathbf{T} = \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H}\mathbf{N}^{-1}$. The inverse is obtained from (2.25) by virtue of $(\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{Q}_{\overline{\mathbf{A}}})^\dagger = \mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}(\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})}\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{P}_{\mathcal{N}(\overline{\mathbf{A}})})^\dagger$, leading to the representation claimed in point (ii) of the theorem. □

From point (i) of Theorem 8 it follows that $\dim[\mathcal{N}(\mathbf{A})\cap\mathcal{N}(\overline{\mathbf{A}})] = r - \mathrm{rk}(\overline{\mathbf{G}}) - \mathrm{rk}(\mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{K}^*)$, whereas from point (ii) it is seen that if $\overline{\mathbf{A}}$ is nonsingular, then it is GR if and only if $\mathbf{Q}_{\overline{\mathbf{G}}^*} = \mathbf{Q}_{\overline{\mathbf{G}}^*}\mathbf{S}^*(\mathbf{S}\mathbf{Q}_{\overline{\mathbf{G}}^*})^\dagger$; see e.g., [16, Exercise 5.10.12].

The theorem below identifies conditions necessary and sufficient for one of the subspaces $\mathcal{R}(\overline{\mathbf{A}})$ and $\mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$ to be contained in the other. Such inclusions, in particular $\mathcal{R}(\overline{\mathbf{A}}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$, play an important role in comparisons of estimators under the general linear model; see e.g., [17, Section 2].

Theorem 9. *Let* $\mathbf{A}$ *be of the form* (1.2)*. Then:*

(i) $\mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) \subseteq \mathcal{R}(\overline{\mathbf{A}})$ *if and only if* $\overline{\mathbf{A}}$ *is properly partitioned and* $\mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma}) \subseteq \mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma K})$,
(ii) $\mathcal{R}(\overline{\mathbf{A}}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$ *if and only if* $\mathcal{R}(\mathbf{\Sigma L}) \subseteq \mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma})$ *and* $\mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma K}) \subseteq \mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma})$.

Proof. From (1.2)–(1.4) it follows that

$$\mathbf{I}_n - \mathbf{A}\mathbf{A}^* = \mathbf{U}\begin{pmatrix} \mathbf{I}_r - \mathbf{\Sigma}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*. \tag{2.26}$$

Since $\mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) \subseteq \mathcal{R}(\overline{\mathbf{A}}) \Leftrightarrow \mathbf{P}_{\overline{\mathbf{A}}}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) = \mathbf{I}_n - \mathbf{A}\mathbf{A}^*$, from (2.11) and (2.26) we conclude that $\mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) \subseteq \mathcal{R}(\overline{\mathbf{A}})$ is equivalent to the conjunction

$$\mathbf{N} = \mathbf{I}_{n-r}, \qquad \mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H} = \mathbf{0}, \quad \text{and} \quad \mathbf{P}_{\overline{\mathbf{G}}}(\mathbf{I}_r - \mathbf{\Sigma}^2) = \mathbf{I}_r - \mathbf{\Sigma}^2. \tag{2.27}$$

However, the first two conditions in (2.27) are equivalent, which means that one of them, say the first one, is redundant. The second condition in (2.27) satisfies $\mathbf{Q}_{\overline{\mathbf{G}}}\mathbf{H} = \mathbf{0} \Leftrightarrow \mathcal{R}(\mathbf{H}) \subseteq \mathcal{R}(\overline{\mathbf{G}})$, whereas the third one fulfils $\mathbf{P}_{\overline{\mathbf{G}}}(\mathbf{I}_r - \mathbf{\Sigma}^2) = \mathbf{I}_r - \mathbf{\Sigma}^2 \Leftrightarrow \mathbf{P}_{\overline{\mathbf{G}}}(\mathbf{I}_r - \mathbf{\Sigma}) = \mathbf{I}_r - \mathbf{\Sigma} \Leftrightarrow \mathcal{R}(\mathbf{I}_r - \mathbf{\Sigma}) \subseteq \mathcal{R}(\overline{\mathbf{G}})$, from where point (i) of the theorem follows.

The proof of the second part is established in a similar fashion. Direct calculations involving (2.1) and (2.26) show that $\mathcal{R}(\overline{\mathbf{A}}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) \Leftrightarrow \mathbf{P}_{\mathbf{I}_n - \mathbf{A}\mathbf{A}^*}\overline{\mathbf{A}} = \overline{\mathbf{A}}$ holds if and only if $\mathbf{P}_{\mathbf{I}_r - \boldsymbol{\Sigma}}\overline{\mathbf{G}} = \overline{\mathbf{G}}$ and $\mathbf{P}_{\mathbf{I}_r - \boldsymbol{\Sigma}}\mathbf{H} = \mathbf{H}$. This conjunction can be equivalently expressed as $\mathcal{R}(\overline{\mathbf{G}}) \subseteq \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma})$ and $\mathcal{R}(\mathbf{H}) \subseteq \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma})$, whence the assertion follows. □

The first observation originating from Theorem 9 concerns the class of contractions. Recall that $\mathbf{A} \in \mathbb{C}_{n,n}$ is called a contraction if the Euclidean norm of $\mathbf{A}\mathbf{x}$ is not greater than the Euclidean norm of $\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}_{n,1}$; see [18, Exercise 6.43]. From [10, Corollary 4.2], it is known that when $\mathbf{A}$ is a contraction, then $\mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) \subseteq \mathcal{R}(\overline{\mathbf{A}})$. Thus, from point (i) of Theorem 9 we conclude that a sufficient condition for $\overline{\mathbf{A}}$ to be properly partitioned is that $\mathbf{A}$ is a contraction. Another characteristics obtained from Theorem 9 is that

$$\mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) = \mathcal{R}(\overline{\mathbf{A}}) \Leftrightarrow \mathcal{R}(\boldsymbol{\Sigma}\mathbf{L}) \subseteq \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma}\mathbf{K}) = \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma}). \tag{2.28}$$

This equivalence leads to the following characterizations of the class of orthogonal projectors.

Theorem 10. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then the following statements are equivalent:*

(i) *$\mathbf{A}$ is an orthogonal projector,*
(ii) *$\mathbf{A}$ is an oblique projector and $\mathcal{R}(\overline{\mathbf{A}}) = \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$,*
(iii) *$\mathbf{A}$ is a partial isometry and $\mathcal{R}(\overline{\mathbf{A}}) = \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$.*

Proof. The proof will be based on the characterizations given below Lemma 1. First recall that $\mathbf{A}$ is an orthogonal projector if and only if $\boldsymbol{\Sigma} = \mathbf{I}_r$ and $\mathbf{K} = \mathbf{I}_r$. Substituting $\boldsymbol{\Sigma}\mathbf{K} = \mathbf{I}_r$, i.e., a condition necessary and sufficient for $\mathbf{A}$ to be an oblique projector, to the right-hand side of the equivalence (2.28) leads to $\mathbf{L} = \mathbf{0}$. Since $\mathbf{L} = \mathbf{0} \Leftrightarrow \mathbf{K}^* = \mathbf{K}^{-1}$, it follows that $\boldsymbol{\Sigma} = \mathbf{I}_r$ and $\mathbf{K} = \mathbf{I}_r$, which establishes the equivalence (i) ⇔ (ii). Similarly, substituting $\boldsymbol{\Sigma} = \mathbf{I}_r$, which is a necessary and sufficient condition for $\mathbf{A}$ to be a partial isometry, to the right-hand side of (2.28) gives $\mathbf{L} = \mathbf{0}$ and $\mathbf{K} = \mathbf{I}_r$ which shows that (i) ⇔ (iii). □

Observe that part (i) ⇔ (ii) of Theorem 10 is a modified version of the first claim in [12, Theorem 10] according to which $\mathbf{A}$ is an orthogonal projector if and only if $\mathbf{A}$ is an oblique projector and $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) = n - r$. It can be verified that $\mathrm{rk}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) = n - r$ if and only if $\mathbf{A}$ is a partial isometry. Further properties of $\mathbf{I}_n - \mathbf{A}\mathbf{A}^*$ are identified in what follows.

Theorem 11. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$. Then $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$ if and only if $\mathbf{I}_n - \mathbf{A}\mathbf{A}^*$ is nonsingular. Moreover, if $\mathbf{A}$ is of the form (1.2), then $\mathcal{R}(\mathbf{A}^*) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$ if and only if $\mathcal{R}(\mathbf{K}^*) \subseteq \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma})$.*

Proof. In view of $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*) \Leftrightarrow \mathbf{P}_{\mathbf{I}_n - \mathbf{A}\mathbf{A}^*}\mathbf{A} = \mathbf{A}$, from (1.2) and (2.26) it is seen that $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$ if and only if $(\mathbf{I}_r - \boldsymbol{\Sigma}^2)(\mathbf{I}_r - \boldsymbol{\Sigma}^2)^\dagger\boldsymbol{\Sigma}\mathbf{K} = \boldsymbol{\Sigma}\mathbf{K}$ and $(\mathbf{I}_r - \boldsymbol{\Sigma}^2)(\mathbf{I}_r - \boldsymbol{\Sigma}^2)^\dagger\boldsymbol{\Sigma}\mathbf{L} = \boldsymbol{\Sigma}\mathbf{L}$. Combining the former of these equalities postmultiplied by $\mathbf{K}^*$ with the latter one postmultiplied by $\mathbf{L}^*$ gives $(\mathbf{I}_r - \boldsymbol{\Sigma}^2)(\mathbf{I}_r - \boldsymbol{\Sigma}^2)^\dagger\boldsymbol{\Sigma} = \boldsymbol{\Sigma}$, or, equivalently, $(\mathbf{I}_r - \boldsymbol{\Sigma}^2)(\mathbf{I}_r - \boldsymbol{\Sigma}^2)^\dagger = \mathbf{I}_r$. Thus, $\mathrm{rk}(\mathbf{I}_r - \boldsymbol{\Sigma}^2) = r$. By virtue of $\mathbf{I}_r - \boldsymbol{\Sigma}^2 = (\mathbf{I}_r - \boldsymbol{\Sigma})(\mathbf{I}_r + \boldsymbol{\Sigma})$, from (2.26) it is seen that nonsingularity of $\mathbf{I}_r - \boldsymbol{\Sigma}$ entails nonsingularity of $\mathbf{I}_n - \mathbf{A}\mathbf{A}^*$. Since the sufficiency part of the first assertion of the theorem is evident, we consider now the second claim given therein. Here, the proof is limited to the observation that using (1.2) and (2.26) shows that $\mathbf{P}_{\mathbf{I}_n - \mathbf{A}\mathbf{A}^*}\mathbf{A}^* = \mathbf{A}^*$ if and only if $\mathcal{R}(\mathbf{K}^*\boldsymbol{\Sigma}) \subseteq \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma})$, which is equivalent to $\mathcal{R}(\mathbf{K}^*) \subseteq \mathcal{R}(\mathbf{I}_r - \boldsymbol{\Sigma})$. □

From Theorem 11, it follows that when $\mathbf{A}$ is nilpotent of index 2, i.e., when $\mathbf{K} = \mathbf{0}$, then $\mathcal{R}(\mathbf{A}^*) \subseteq \mathcal{R}(\mathbf{I}_n - \mathbf{A}\mathbf{A}^*)$ necessarily holds. Moreover, this inclusion is also satisfied when $\mathbf{I}_r - \boldsymbol{\Sigma}$ is nonsingular, which happens when all singular values of $\mathbf{A}$ are different from 1.

Hartwig and Spindelböck [10, Lemma 3.3] have shown that when $\mathbf{A}$ is a contraction, then $\mathbf{I}_n - \mathbf{A}$ is EP. In the next theorem, this result is extended to an equivalence.

Theorem 12. *Let $\mathbf{A} \in \mathbb{C}_{n,n}$ be such that $\mathbf{A}^2 = \mathbf{A}$. Then the following statements are equivalent:*

(i) *$\overline{\mathbf{A}}$ is Hermitian,*
(ii) *$\overline{\mathbf{A}}$ is EP,*
(iii) *$\mathbf{A}$ is a contraction.*

Proof. As already mentioned, the implication (iii) ⇒ (ii) follows from [10, Lemma 3.3]. From the characteristics given below Lemma 1, we conclude that when an idempotent matrix is EP, then it is necessary Hermitian, from where the part (ii) ⇒ (i) is derived. Finally, the implication (i) ⇒ (iii) reflects a known fact that every orthogonal projector is a contraction. □

The paper is concluded with some remarks indicating that the decomposition provided in Lemma 1 is useful not only to deal with $\mathbf{A}$ and $\overline{\mathbf{A}}$, and can be exploited to investigate also properties of more involved functions of $\mathbf{A}$. Consider $\mathbf{A} - \lambda\mathbf{I}_n$, where $\lambda \in \mathbb{C}$ is nonzero. A counterpart of the representation (1.2) for $\lambda^{-1}\mathbf{A}$ is

$$\lambda^{-1}\mathbf{A} = \mathbf{U}\begin{pmatrix} \boldsymbol{\Sigma}_\lambda\mathbf{K} & \boldsymbol{\Sigma}_\lambda\mathbf{L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{2.29}$$

where $\boldsymbol{\Sigma}_\lambda = |\lambda|^{-1}\boldsymbol{\Sigma}$. This fact can be based on the following two observations: (i) since the nonzero eigenvalues of $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{A}$ are $\sigma_j^2, j = 1, \ldots, r$, the nonzero eigenvalues of $\lambda^{-1}\mathbf{A}(\lambda^{-1}\mathbf{A})^*$ and $(\lambda^{-1}\mathbf{A})^*\lambda^{-1}\mathbf{A}$ are $(\sigma_j/|\lambda|)^2$, (ii) $\mathbf{A}\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{A}$ have the same eigenvectors as $\lambda^{-1}\mathbf{A}(\lambda^{-1}\mathbf{A})^*$ and $(\lambda^{-1}\mathbf{A})^*\lambda^{-1}\mathbf{A}$, respectively.

Several characteristics of the matrix $\mathbf{A} - \lambda\mathbf{I}_n$ can be obtained directly from the results established above. For example, an illuminating illustration of the applicability of the representation (2.29) is the following original solution to part (a) of Problem 5.11.14 in [16], which asserts that if $\mathbf{A}$ is normal, then $\mathcal{R}(\mathbf{A} - \lambda\mathbf{I}_n)$ is perpendicular to $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I}_n)$ for every scalar λ. For $\lambda = 0$, the claim is visibly fulfilled. To show the requested property when λ is nonzero, first observe that normality of $\mathbf{A}$, i.e., $\mathbf{L} = \mathbf{0}$ and $\mathbf{K\Sigma} = \mathbf{\Sigma K}$, ensures that $\mathbf{G}_\lambda = \mathbf{\Sigma}_\lambda\mathbf{K}$ is normal as well. On the other hand, $\mathcal{R}(\mathbf{A} - \lambda\mathbf{I}_n) \perp \mathcal{N}(\mathbf{A} - \lambda\mathbf{I}_n)$ is equivalent to $\mathbf{P}_{\mathbf{A}-\lambda\mathbf{I}_n}\mathbf{P}_{\mathcal{N}(\mathbf{A}-\lambda\mathbf{I}_n)} = \mathbf{0}$. Since for nonzero λ, we have $\mathcal{R}(\mathbf{A}-\lambda\mathbf{I}_n) = \mathcal{R}(\mathbf{I}_n-\lambda^{-1}\mathbf{A})$ and $\mathcal{N}(\mathbf{A}-\lambda\mathbf{I}_n) = \mathcal{N}(\mathbf{I}_n-\lambda^{-1}\mathbf{A})$ (see [16, p. 178]), from (2.11) and (2.15) it follows that

$$\mathbf{P}_{\mathbf{A}-\lambda\mathbf{I}_n} = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{G}}_\lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r}\end{pmatrix}\mathbf{U}^* \quad \text{and} \quad \mathbf{P}_{\mathcal{N}(\mathbf{A}-\lambda\mathbf{I}_n)} = \mathbf{U}\begin{pmatrix}\mathbf{Q}_{\overline{\mathbf{G}}_\lambda^*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*. \tag{2.30}$$

In consequence, $\mathcal{R}(\mathbf{A} - \lambda\mathbf{I}_n) \perp \mathcal{N}(\mathbf{A} - \lambda\mathbf{I}_n)$ if and only if $\mathbf{P}_{\overline{\mathbf{G}}_\lambda}\mathbf{Q}_{\overline{\mathbf{G}}_\lambda^*} = \mathbf{0}$. This equality is satisfied, by the fact that when $\mathbf{G}_\lambda$ is normal, so is $\overline{\mathbf{G}}_\lambda$, and $\overline{\mathbf{G}}_\lambda$ is EP, i.e., $\mathbf{P}_{\overline{\mathbf{G}}_\lambda} = \mathbf{P}_{\overline{\mathbf{G}}_\lambda^*}$.

Note that the right-hand side expression in (2.30) may be useful in considerations involving eigenspaces of square matrices.

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